Uncertainty Relations in the Presence of Quantum Memory for Mutually Unbiased Measurements

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Joint work with Nan Wu and Fangmin Song (arXiv:1807.01047)

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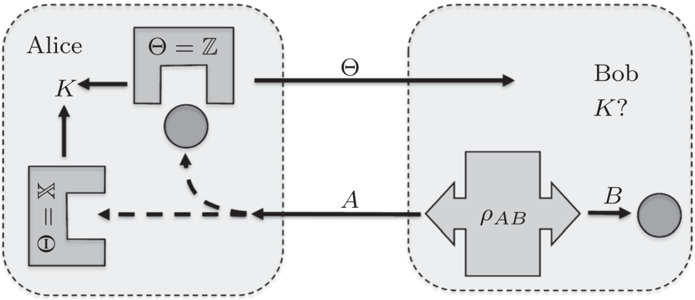
Outline

Guessing game



- A guessing game[1] played by Alice and Bob
  1. Bob prepares state ρ_{AB} and sends ρ_A to Alice
  2. Alice measures either $\mathbb{X}$ or $\mathbb{Z}$ (uniformly) and stores outcome K
  3. Alice tells Bob which measurement Θ has been conducted
  4. Bob guesses the value of K
- $\mathbb{X}$ and $\mathbb{Z}$ are known to both Alice and Bob
- How well can Bob guess K on average?

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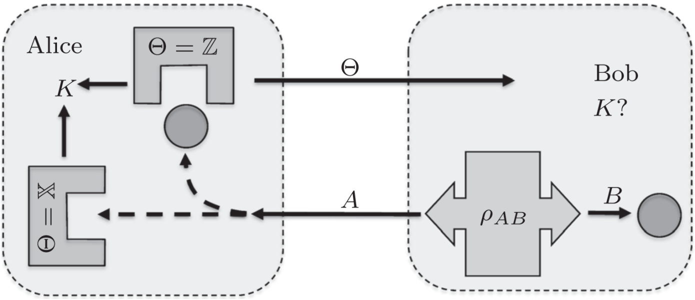

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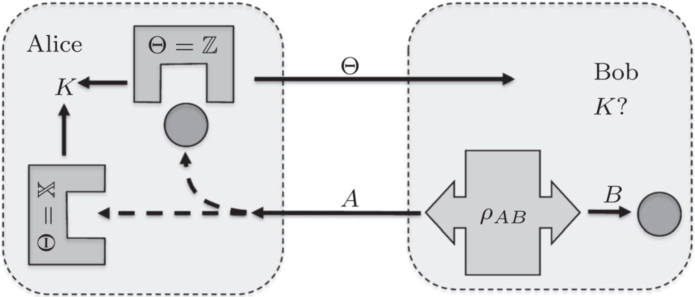

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Entanglement helps!

- Assume $\mathbb{X} = \{|x_i\rangle\}$ and $\mathbb{Z} = \{|z_j\rangle\}$ be complementary on A: $|\langle x_i|z_j\rangle| = 1/\sqrt{d}$ for arbitrary i, j
- Suppose Bob prepares a maximally entangled state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_i |x_i\rangle_A |x_i\rangle_B = \frac{1}{\sqrt{d}} \sum_j |z_j\rangle_A |z_j\rangle_B$$

  and sends ρ_A to Alice
- Measuring ρ_A with $\mathbb{X}/\mathbb{Z}$, Alice gets classical-quantum (cq.) states

$$\omega_{XB} = \frac{1}{d} \sum_i |x_i\rangle\langle x_i|_A \otimes |x_i\rangle\langle x_i|_B$$

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- If Alice obtains x_i, Bob processes state x_i. Similar for $\mathbb{Z}$
- Bob can guess K with certainty by measuring his state
- **Entanglement** reduces the Bob's **uncertainty** about K

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Uncertainty relation (in the presence of memory)

- How much can the entanglement reduce uncertainty?
- That is, what if Bob prepares an arbitrary state ρ_{AB}?
- Measuring ρ_A with $\mathbb{X}/\mathbb{Z}$, Alice gets two cq. states

$$\omega_{XB} = \sum_i p_i |x_i\rangle\langle x_i| \otimes \omega_i^B, \quad p_i = \mathrm{Tr}\langle x_i|\rho_{AB}|x_i\rangle,$$

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- To answer this question, we must know
  1. How to quantify the entanglement of ρ_{AB}?
  2. How to quantify the uncertainty of ω_{XB} and τ_{ZB}?
- Uncertainty relation in the presence of memory (UR)[2]

$$\mathrm{H}(X|B)_\omega + \mathrm{H}(Z|B)_\tau \geqslant \log d + \mathrm{H}(A|B)_\rho$$

- $\mathrm{H}(A|B)_{\omega,\tau,\rho}$ is the conditional entropy of state ω, τ, and ρ
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Ingredients of a uncertainty relation

$$\mathrm{H}(X|B)_\omega + \mathrm{H}(Z|B)_\tau \geqslant \log d + \mathrm{H}(A|B)_\rho$$

- Five ingredients of a general uncertainty relation

  Incompatible measurements: $\mathbb{X}$ and $\mathbb{Z}$
  State being measured: bipartite state ρ_{AB}
  Uncertainty measure: conditional entropy $\mathrm{H}(A|B)_{\omega,\tau}$
  Uncertainty relation form: lower bound on sum of uncertainties
  Entanglement measure: conditional entropy $\mathrm{H}(A|B)_\rho$

- Our relation

  Incompatible measurements: complete set of mutually unbiased measurements
  State being measured: bipartite state ρ_{AB}
  Uncertainty measure: conditional collision entropy $\mathrm{H}_2(A|B)_{\omega,\tau}$
  Uncertainty relation form: an equality
  Entanglement measure: conditional collision entropy $\mathrm{H}_2(A|B)_\rho$

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Mutually unbiased measurements

- Let $\mathcal{P}^{(1)} = \{P_x^{(1)}\}_{x\in[d]}$ and $\mathcal{P}^{(2)} = \{P_x^{(2)}\}_{x\in[d]}$ be two POVMs:

$$\forall\theta = 1,2, \quad P_x^{(\theta)} \geqslant 0, \quad \sum_x P_x^{(\theta)} = \mathbb{1}$$

- They are mutually unbiased[3] if for all $x, x' \in [d], \theta = 1, 2$

$$\mathrm{Tr}\left[P_x^{(\theta)}\right] = 1, \quad \textit{each operator is normalized}$$

$$\mathrm{Tr}\left[P_x^{(1)} P_x^{(2)}\right] = \frac{1}{d}, \quad \textit{two measurements are unbiased}$$

$$\mathrm{Tr}\left[P_x^{(\theta)} P_{x'}^{(\theta)}\right] = \delta_{x,x'}\kappa + (1-\delta_{x,x'})\frac{1-\kappa}{d-1}.$$

- The *efficiency parameter* κ satisfies $1/d < \kappa \leqslant 1$
- $\{\mathcal{P}^{(\theta)}\}_{\theta\in\Theta}$ forms a set of MUMs if they are pairwise unbiased
- A complete set of MUMs is a set of MUMs of size $d+1$
- A complete set of MUMs can be explicitly constructed[3]

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Conditional collision entropy

- Let ρ_{AB} be a quantum state on system AB
- The conditional collision entropy is defined as[4]

$$\mathrm{H}_2(A|B)_\rho = -\log \mathrm{Tr}\left[\rho_{AB}(\mathbb{1}_A \otimes \rho_B)^{-1/2}\rho_{AB}(\mathbb{1}_A \otimes \rho_B)^{-1/2}\right]$$

- $\mathbb{1}_A$ is the identity operator
- $-\log d \leqslant \mathrm{H}_2(A|B)_\rho \leqslant \log d$
- For separable states σ_{AB}, $\mathrm{H}_2(A|B)_\sigma \geqslant 0$[5]
- $\mathrm{H}_2(A|B)_\rho < 0 \Rightarrow \rho_{AB}$ must be entangled
- Trivializing system B ($\rho_B = 1$), we get the collision entropy

$$\mathrm{H}_2\left(A\right)_\rho = -\log \mathrm{Tr}\,\rho_A^2$$

[4] Marco Tomamichel. *Quantum Information Processing with Finite Resources: Mathematical Foundations*. Vol. 5. Springer, 2015.

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- $\mathbb{1}_A$ is the identity operator
- $-\log d \leqslant \mathrm{H}_2(A|B)_\rho \leqslant \log d$
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- $\mathrm{H}_2(A|B)_\rho < 0 \Rightarrow \rho_{AB}$ must be entangled
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Post-measurement state for a MUM

- Let $\mathcal{P}^{(\theta)} = \{P_x^{(\theta)}\}_{x\in[d]}$ be a MUM in A
- Measuring ρ_{AB} on A by $\mathcal{P}^{(\theta)}$, we get a cq. state

$$\omega_{X^{(\theta)}B} = \sum_{x=1}^{d} |x\rangle\langle x|_X \otimes \mathrm{Tr}_A\left[\left(P_x^{(\theta)} \otimes \mathbb{1}_B\right)\rho_{AB}\right] \qquad (1)$$

- Register X stores the measurement outcome
- $\mathrm{Tr}_A[(P_x^{(\theta)} \otimes \mathbb{1}_B)\rho_{AB}]$ is the post-measurement state (unnormalized) left on system B
- $\mathrm{Tr}[(P_x^{(\theta)} \otimes \mathbb{1}_B)\rho_{AB}]$ is probability that the outcome is x

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Post-measurement state for complete set of MUMs

- Let $\{\mathcal{P}^{(\theta)}\}_{\theta\in[d+1]}$ be a complete set of MUMs on system A
- Define the following cq. state

$$\omega_{XB\Theta} = \frac{1}{d+1}\sum_{\theta=1}^{d+1}\sum_{x=1}^{d} |x\rangle\langle x|_X \otimes \mathrm{Tr}_A\left[\left(P_x^{(\theta)}\otimes \mathbb{1}_B\right)\rho_{AB}\right]\otimes|\theta\rangle\langle\theta|_\Theta \quad (2)$$

- Θ indicates which MUM has been performed
- $\omega_{XB\Theta}$ is a uniform mixing of $\omega_{X^{(\theta)}B}$: $\omega_{XB\Theta=\theta} = \omega_{X^{(\theta)}B}$
- Conditional collision entropy of $\omega_{XB\Theta}$, with partition $X{:}B\Theta$

$$\mathrm{H}_2\left(X|B\Theta\right)_\omega = -\log\left(\frac{1}{d+1}\sum_{\theta,x}\mathrm{Tr}_B\left\{\mathrm{Tr}_A[P_x^{(\theta)}\widetilde{\rho}_{AB}]^2\right\}\right)$$

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Theorem (An equality relation for complete set of MUMs)

Let $\{\mathcal{P}^{(\theta)}\}_{\theta\in[d+1]}$ *be a complete set of MUMs on system* A*. For arbitrary quantum state* ρ_{AB}*, it holds that*

$$\mathrm{H}_2\left(A|B\Theta\right)_\omega = \log\left(d+1\right) - \log\left(f(\kappa) + g(\kappa)2^{-\,\mathrm{H}_2(A|B)_\rho}\right), \quad (3)$$

where $\omega_{XB\Theta}$ *is defined in Eq.* (2)*, and the coefficients are given by*

$$f(\kappa) = 1 + \frac{1-\kappa}{d-1}, \quad g(\kappa) = \frac{\kappa d - 1}{d-1}.$$

- When $\kappa = 1$, Eq. (3) recovers the main result of [2][6]

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Outline

Pretty-good state discrimination

- **State discrimination:** Let $\mathcal{S} = \{p_i, \rho_i\}$ be a state ensemble. Sample σ from $\mathcal{S}$. What is the index i of σ?
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Pretty-good guessing for a complete set of MUMs

- Each MUM induces a cq. state of the form

$$\omega_{X^{(\theta)}B} = \sum_{x=1}^{d} |x\rangle\langle x| \otimes \mathrm{Tr}_A \left[\left(P_x^{(\theta)} \otimes \mathbb{1}_B \right) \rho_{AB} \right]$$

- How well can Bob guess x?
  - He can guess "pretty-good": $\mathrm{P}^{\mathrm{pg}}(X^{(\theta)}|B)_\omega$
- How well can Bob guess x for a complete set of MUMs, on average?
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Lemma

Let $\{\mathcal{P}^{(\theta)}\}_{\theta \in [d+1]}$ be a complete set of MUMs on system A. For arbitrary quantum state ρ_{AB}, it holds that

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Lower bound on sum of uncertainties

- Uncertainty relations are commonly expressed as lower bound on the sum of uncertainties

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- This is a uncertainty relation without memory
- Recovers a special case ($\alpha = 2$) of **Proposition 3** in [9][10]

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An entanglement detection method

- Let $\{\mathcal{P}^{(\theta)}\}_{\theta\in[d+1]}$ be a complete set of MUMs on A
- Let $\{\mathcal{Q}^{(\theta)}\}_{\theta\in[d+1]}$ be an arbitrary set of $d+1$ measurements on B
- If Alice performs $\mathcal{P}^{(\theta)}$, Bob performs $\mathcal{Q}^{(\theta)}$. They get

$$\omega_{X^{(\theta)}Y^{(\theta)}} = \sum_{x,y=1}^{d} \mathrm{Tr}\left[\left(P_x^{(\theta)} \otimes Q_y^{(\theta)}\right)\rho_{AB}\right] |x\rangle\langle x| \otimes |y\rangle\langle y|.$$

- $\omega_{X^{(\theta)}Y^{(\theta)}}$ can be evaluated from measurement statistics

Lemma

For arbitrary separable quantum state ρ_{AB}, it holds that

$$\frac{1}{d+1}\sum_{\theta=1}^{d+1} \mathrm{H}_2\left(X^{(\theta)}\middle|Y^{(\theta)}\right)_\omega \geqslant \log(d+1) - \log\left(f(\kappa)+g(\kappa)\right).$$

An entanglement detection method (cont.)

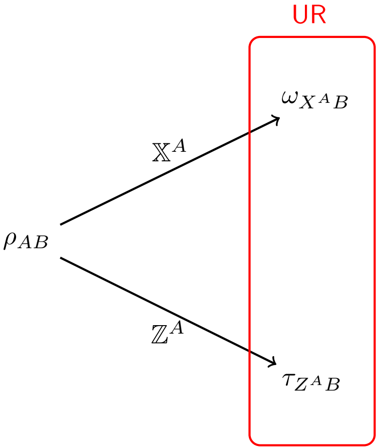
- How does the detection method work?
- Suppose now there exists a source producing states ρ_{AB}
- Alice and Bob sample from the source and gather statistics
- They estimate the joint distribution for each pair $\{\mathcal{P}^{(\theta)}, \mathcal{Q}^{(\theta)}\}$
- They evaluate the sum of (classical) conditional collision entropies
- According to the above lemma, the source is entangled if

$$\frac{1}{d+1}\sum_{\theta=1}^{d+1} \mathrm{H}_2\left(X^{(\theta)}\middle|Y^{(\theta)}\right)_{\omega} < \log\left(d+1\right) - \log\left(f(\kappa)+g(\kappa)\right). \quad (5)$$

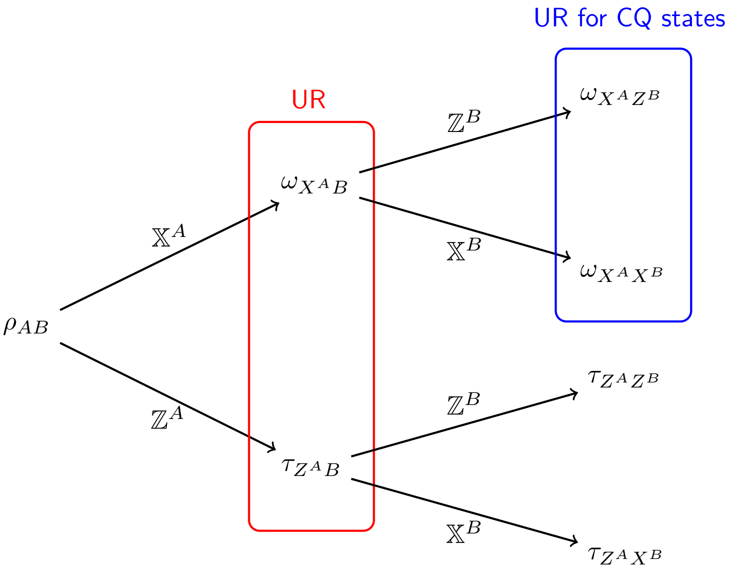
- The choice of measurements $\{\mathcal{Q}^{(\theta)}\}$ on system B is arbitrary
- For best detection criterion, minimize the LHS. of Eq. (5) by optimizing over all possible measurements $\{\mathcal{Q}^{(\theta)}\}$

Outline

A unified view



A unified view (cont.)



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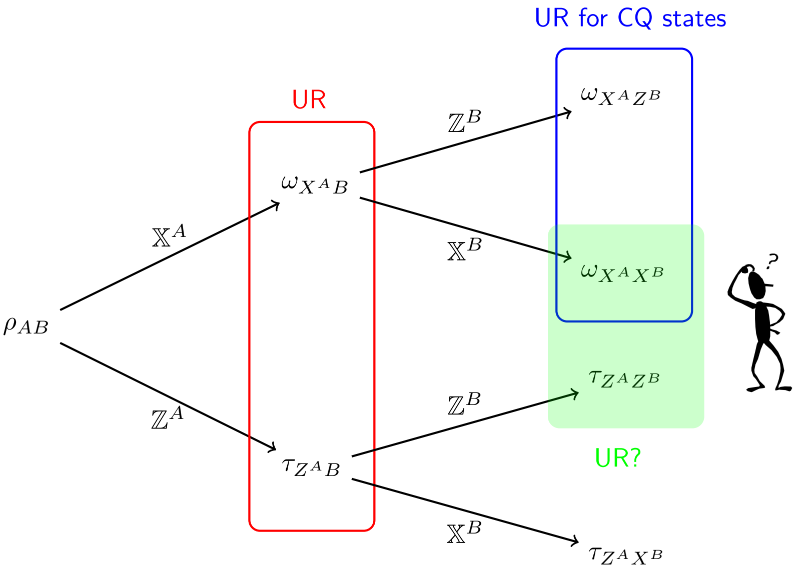

CQC conjecture

- Let $\mathbb{X}^A$ and $\mathbb{Z}^A$ be complementary on A
- Let $\mathbb{X}^B$ and $\mathbb{Z}^B$ be complementary on B
- $\mathbb{X}^A \otimes \mathbb{X}^B$ and $\mathbb{Z}^A \otimes \mathbb{Z}^B$ induce two classical states

$$\begin{aligned}
\omega_{X^A X^B} &= \sum_{ij} p_{ij} |x_i^A\rangle\langle x_i^A| \otimes |x_j^B\rangle\langle x_j^B| \\
p_{ij} &= \langle x_i^A x_j^B | \rho_{AB} | x_i^A x_j^B \rangle \\
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- The complementary-quantum correlation conjecture (CQC)[11]

$$\mathrm{I}\left(X^A{:}X^B\right)_\omega + \mathrm{I}\left(Z^A{:}Z^B\right)_\tau \leqslant \mathrm{I}(A{:}B)_\rho$$

- $\mathrm{I}(A{:}B)$ is the mutual information quantifying the correlation between A and B

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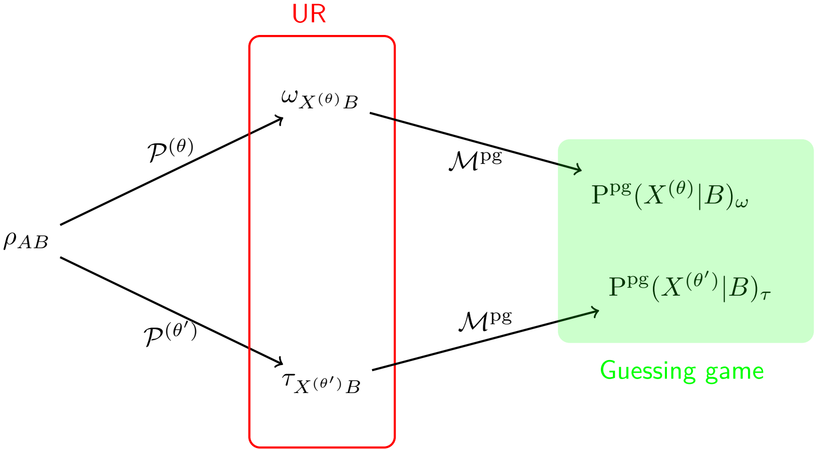
Our work fits into the unified view



Summary

- **What have done?**
  - An equality relation for complete set of MUMs
  - Conditional collision entropy as uncertainty measure
  - Some corollaries from the equality relation
    1. Bound on pretty-good guessing probabilities
    2. An uncertainty relation expressed as sum of uncertainties
    3. An entanglement detection method
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Thank you !

Any questions ?

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